

# On the local structure of noncommutative deformations

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## Abstract

Let  $(M, \pi, \mathcal{D})$  be a Poisson manifold endowed with a flat, torsion-free contravariant connection. We show that if  $\mathcal{D}$  is an  $\mathcal{F}$ -connection then there exists a tensor  $\mathbf{T}$  such that  $\mathcal{D}\mathbf{T}$  is the metacurvature tensor introduced by E. Hawkins in his work on noncommutative deformations. We compute  $\mathbf{T}$  and the metacurvature tensor in this case, and show that if  $\mathbf{T} = 0$  then, near any regular point,  $\pi$  and  $\mathcal{D}$  are defined in a natural way by a Lie algebra action and a solution of the classical Yang-Baxter equation. Moreover, when  $\mathcal{D}$  is the contravariant Levi-Civita connection associated to  $\pi$  and a Riemannian metric, the Lie algebra action preserves the metric.

*Keywords:* Contravariant connections, Metacurvature, Noncommutative deformations.

*2000 MSC:* 53D17, Primary 58B34.

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## 1. Introduction and main result

In [4, 5], Hawkins showed that if a deformation of the graded algebra  $\Omega^*(M)$  of differential forms on a Riemannian manifold  $M$  comes from a spectral triple describing  $M$ , then the Poisson tensor  $\pi$  (which characterizes the deformation) and the Riemannian metric satisfy the following conditions:

- ( $H_1$ ) The associated metric contravariante connexion  $\mathcal{D}$  is flat.
- ( $H_2$ ) The metacurvature of  $\mathcal{D}$  vanishes.
- ( $H_3$ ) The Poisson tensor  $\pi$  is compatible with the Riemannian volume  $\mu$ :

$$d(i_\pi \mu) = 0.$$

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The metric contravariant connection associated naturally to any pair of pseudo-Riemannian metric and Poisson tensor is the contravariant analogue of the classical Levi-Civita connection; it has appeared first in [9]. The metacurvature, introduced in [5], is a (2,3)-tensor field (symmetric in the contravariant indices and antisymmetric in the covariant indices) associated naturally to any flat, torsion-free contravariant connection.

The main result of Hawkins (cf. [5]) states that if  $(M, \pi, g)$  is a triple satisfying  $(H_1) - (H_3)$  with  $M$  compact, then around any regular point  $x_0 \in M$  the Poisson tensor can be written as

$$\pi = \sum_{i,j} a_{ij} X_i \wedge X_j \quad (1)$$

where  $a_{ij}$  are constants and  $\{X_1, \dots, X_s\}$  is a family of linearly independent commuting Killing vector fields.

On the other hand, the first author showed in [11] that if  $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}^1(M)$  is an action of a finite-dimensional real Lie algebra  $\mathfrak{g}$  on a smooth manifold  $M$ , and  $r \in \wedge^2 \mathfrak{g}$  is a solution of the classical Yang-Baxter equation, then the map  $\mathcal{D}^r : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M)$  given by

$$\mathcal{D}_\alpha^r \beta := \sum_{i,j=1}^n a_{ij} \alpha(\zeta(u_i)) L_{\zeta(u_j)} \beta \quad (2)$$

where  $\{u_1, \dots, u_n\}$  is any basis of  $\mathfrak{g}$  and  $a_{ij}$  are the components of  $r$  in this basis, depends only on  $r$  and  $\zeta$  and defines a flat, torsion-free contravariant connection with respect to the Poisson tensor  $\pi^r := \zeta(r)$ . Moreover, if  $M$  is Riemannian, then  $\mathcal{D}^r$  is nothing else but the metric contravariant connection associated to the metric and  $\pi^r$ , provided that the action preserves the metric. He also showed that when  $\mathfrak{g}$  acts freely on  $M$ , the metacurvature of  $\mathcal{D}^r$  vanishes.

In this setting, (1) can be reexpressed by saying that there exists a free action  $\xi : \mathfrak{g} \rightarrow \mathfrak{X}^1(U)$  of a finite-dimensional abelian Lie algebra  $\mathfrak{g}$  on an open neighborhood  $U$  of  $x_0$ , which preserves  $g$ , and a solution  $r \in \wedge^2 \mathfrak{g}$  of the classical Yang-Baxter equation such that  $\pi = \pi^r$ . Moreover, since  $\xi$  preserves  $g$ ,  $\mathcal{D} = \mathcal{D}^r$  where  $\mathcal{D}$  is the metric contravariant connection associated to  $\pi$  and  $g$ . Therefore,  $\mathcal{D}$  is a Poisson connection, i.e.  $\mathcal{D}\pi = 0$ , and hence an  $\mathcal{F}^{reg}$ -connection (see [10]).

Given a flat, torsion-free  $\mathcal{F}^{reg}$ -connection  $\mathcal{D}$  on a Poisson manifold  $(M, \pi)$ , we shall see that there exists a (2,2)-type tensor field  $\mathbf{T}$  on the dense open set of regular points such that

- (i)  $\mathcal{D}\mathbf{T} = \mathcal{M}$  where  $\mathcal{M}$  is the metacurvature of  $\mathcal{D}$ ;
- (ii)  $\mathbf{T}$  vanishes if and only if the exterior differential of any parallel 1-form is also parallel.

By looking at the proof of Boucetta's result closely, one observes that in order to show that the metacurvature vanishes when the action is free, the first author shows, in fact, that  $\mathcal{D}^r$  is an  $\mathcal{F}^{reg}$ -connection and that whenever a 1-form is  $\mathcal{D}^r$ -parallel then so is its exterior differential, meaning that  $\mathbf{T}$  vanishes (and hence so does the metacurvature). In the case studied by Hawkins  $\mathbf{T}$  vanishes since as we saw above the Lie algebra action  $\xi$  is free. So it is natural to study the following problem, inverse of Boucetta's result: Given a triple  $(M, \pi, g)$  whose metric contravariant connection is a flat  $\mathcal{F}^{reg}$ -connection and such that  $\mathbf{T} = 0$ , is there a free action of a finite-dimensional Lie algebra  $\mathfrak{g}$  preserving  $g$  and a solution  $r \in \wedge^2 \mathfrak{g}$  of the classical Yang-Baxter equation such that  $\pi = \pi^r$  and  $\mathcal{D} = \mathcal{D}^r$ ?

The main result of this paper gives a positive answer to this question in a more general setting. More precisely,

**Theorem 1.1.** *Let  $(M, \pi, \mathcal{D})$  be a Poisson manifold endowed with a flat, torsion-free contravariant connection.*

- (1) *If  $\mathcal{D}$  is an  $\mathcal{F}^{reg}$ -connection and  $\mathbf{T} = 0$ , then for any regular point  $x_0$  with rank  $2r$ , there exists a free action  $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}(U)$  of a  $2r$ -dimensional real Lie algebra  $\mathfrak{g}$  on neighborhood  $U$  of  $x_0$ , and an invertible solution  $r \in \wedge^2 \mathfrak{g}$  of the classical Yang-Baxter equation, such that  $\pi = \pi^r$  and  $\mathcal{D} = \mathcal{D}^r$ .*
- (2) *Moreover, if  $\mathcal{D}$  is the metric contravariant connection associated to  $\pi$  and a Riemannian metric  $g$ , then the action can be chosen in such a way that its fundamental vector fields are Killing.*

The paper is organized as follows. In Section 2, we recall some standard facts about contravariant connections and the metacurvature tensor; we also define the tensor  $\mathbf{T}$ . Section 3 is devoted to the computation of the metacurvature tensor (and the tensor  $\mathbf{T}$  as well) in the case of an  $\mathcal{F}^{reg}$ -connection. In the last section, we give a proof of Theorem 1.1.

**Notation 1.2.** For a smooth manifold  $M$ ,  $\mathcal{C}^\infty(M)$  will denote the space of smooth functions on  $M$ ,  $\Gamma(V)$  will denote the space of smooth sections of a vector bundle  $V$  over  $M$ ,  $\Omega^p(M) := \Gamma(\wedge^p T^*M)$  will denote the space

of differential  $p$ -forms, and  $\mathfrak{X}^p(M) := \Gamma(\wedge^p TM)$  will denote the space of  $p$ -vector fields.

For a Poisson tensor  $\pi$  on  $M$ , we will denote by  $\pi_\sharp : T^*M \rightarrow TM$  the anchor map defined by  $\beta(\pi_\sharp(\alpha)) = \pi(\alpha, \beta)$ , and by  $H_f$  the Hamiltonian vector field of a function  $f$ , that is,  $H_f := \pi_\sharp(df)$ . We will also denote by  $[\ , \ ]_\pi$  the Koszul-Schouten bracket on differential forms (see, e.g., [8]); this is given on 1-forms by

$$[\alpha, \beta]_\pi = L_{\pi_\sharp(\alpha)}\beta - L_{\pi_\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

The symplectic foliation of  $(M, \pi)$  will be denoted by  $\mathcal{S}$ , and  $T\mathcal{S} = \text{Im } \pi_\sharp$  will be its associated tangent distribution. Finally, we will denote by  $M^{reg}$  the dense open set where the rank of  $\pi$  is locally constant.

## 2. Preliminaries

### 2.1. Contravariant connections

Contravariant connections on Poisson manifolds were defined by Vaismann [7] and studied in detail by Fernandes [14]. These connections play an important role in Poisson geometry (see for instance [14, 13]) and have recently turned out to be useful in other branches of mathematics (e.g., [4, 5]).

The definition of a contravariant connection mimics the usual definition of a covariant connection, except that cotangent vectors have taken the place of tangent vectors. More precisely, a *contravariant connection* on a Poisson manifold  $(M, \pi)$  is an  $\mathbb{R}$ -bilinear map

$$\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M), (\alpha, \beta) \mapsto \mathcal{D}_\alpha \beta$$

such that for any  $f \in \mathcal{C}^\infty(M)$ ,

$$\mathcal{D}_{f\alpha}\beta = f\mathcal{D}_\alpha\beta \quad \text{and} \quad \mathcal{D}_\alpha(f\beta) = f\mathcal{D}_\alpha\beta + \pi_\sharp(\alpha)(f)\beta.$$

A contravariant connection  $\mathcal{D}$  is called an  $\mathcal{F}$ -connection [14] if it satisfies

$$(\forall a \in T^*M, \pi_\sharp(a) = 0) \implies \mathcal{D}_a = 0.$$

We call  $\mathcal{D}$  an  $\mathcal{F}^{reg}$ -connection if the restriction of  $\mathcal{D}$  to  $M^{reg}$  is an  $\mathcal{F}$ -connection.

The *torsion* and the *curvature* of a contravariant connection  $\mathcal{D}$  are formally identical to the usual ones:

$$\begin{aligned} T(\alpha, \beta) &= \mathcal{D}_\alpha \beta - \mathcal{D}_\beta \alpha - [\alpha, \beta]_\pi, \\ R(\alpha, \beta)\gamma &= \mathcal{D}_\alpha \mathcal{D}_\beta \gamma - \mathcal{D}_\beta \mathcal{D}_\alpha \gamma - \mathcal{D}_{[\alpha, \beta]_\pi} \gamma. \end{aligned}$$

These are (2,1) and (3,1)-type tensor fields, respectively. When  $T \equiv 0$  (resp.  $R \equiv 0$ ),  $\mathcal{D}$  is called *torsion-free* (resp. *flat*). In local coordinates  $(x_1, \dots, x_d)$ , the local components of the torsion and curvature tensor fields are given by

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k - \frac{\partial \pi_{ij}}{\partial x_k}, \quad (3)$$

$$R_{ijk}^l = \sum_{m=1}^d \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m + \pi_{im} \frac{\partial \Gamma_{jk}^l}{\partial x_m} - \pi_{jm} \frac{\partial \Gamma_{ik}^l}{\partial x_m} - \frac{\partial \pi_{ij}}{\partial x_m} \Gamma_{mk}^l, \quad (4)$$

where  $\Gamma_{ij}^k$  are the *Christoffel symbols* defined by:  $\mathcal{D}_{dx_i} dx_j = \sum_{k=1}^d \Gamma_{ij}^k dx_k$ .

Given a (pseudo-)Riemannian metric  $g$  on a Poisson manifold  $(M, \pi)$ , one has a contravariant version of the Levi-Civita connection: there exists a unique torsion-free contravariant connection  $\mathcal{D}$  on  $M$  which is metric-compatible, i.e.,

$$\pi_\#(\alpha) \cdot \langle \beta, \gamma \rangle = \langle \mathcal{D}_\alpha \beta, \gamma \rangle + \langle \beta, \mathcal{D}_\alpha \gamma \rangle \quad \forall \alpha, \beta, \gamma \in \Omega^1(M),$$

where  $\langle \cdot, \cdot \rangle$  denotes the metric pairing induced by  $g$ . This connection is determined by the formula:

$$\begin{aligned} \langle \mathcal{D}_\alpha \beta, \gamma \rangle &= \frac{1}{2} \left\{ \pi_\#(\alpha) \cdot \langle \beta, \gamma \rangle + \pi_\#(\beta) \cdot \langle \alpha, \gamma \rangle - \pi_\#(\gamma) \cdot \langle \alpha, \beta \rangle \right. \\ &\quad \left. + \langle [\alpha, \beta]_\pi, \gamma \rangle - \langle [\beta, \gamma]_\pi, \alpha \rangle + \langle [\gamma, \alpha]_\pi, \beta \rangle \right\}, \end{aligned} \quad (5)$$

and is called the *metric contravariant connection* (or *contravariant Levi-Civita connection*) associated to  $(\pi, g)$ .

## 2.2. The metacurvature

In this subsection we recall briefly from [5] the definition of the metacurvature tensor and give some related formulas.

Let  $(M, \pi)$  be a Poisson manifold. Given a torsion-free contravariant connection  $\mathcal{D}$  on  $M$ , there exists a unique bracket  $\{ \cdot, \cdot \}$  on the space  $\Omega^*(M)$  of differential forms, with the following properties:

1.  $\{ , \}$  is bilinear, degree 0 and antisymmetric

$$\{\sigma, \tau\} = -(-1)^{\deg(\sigma)\deg(\tau)}\{\tau, \sigma\}. \quad (6)$$

2.  $\{ , \}$  satisfies the product rule

$$\{\sigma, \tau \wedge \rho\} = \{\sigma, \tau\} \wedge \rho + (-1)^{\deg(\sigma)\deg(\tau)}\tau \wedge \{\sigma, \rho\}. \quad (7)$$

3. The exterior differential  $d$  is a derivation with respect to  $\{ , \}$ , i.e.,

$$d\{\sigma, \tau\} = \{d\sigma, \tau\} + (-1)^{\deg(\sigma)}\{\sigma, d\tau\}. \quad (8)$$

4. For any  $f, g \in \mathcal{C}^\infty(M)$  and any  $\sigma \in \mathcal{C}^\infty(M)$ ,

$$\{f, g\} = \pi(df, dg) \quad \text{and} \quad \{f, \sigma\} = \mathcal{D}_f \sigma. \quad (9)$$

This bracket is given (on decomposable forms) by

$$\begin{aligned} \{\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_l\} &= (-1)^{k+1} \sum_{i,j} (-1)^{i+j} \{\alpha_i, \beta_j\} \wedge \\ &\quad \wedge \alpha_1 \wedge \cdots \wedge \widehat{\alpha}_i \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \widehat{\beta}_j \wedge \cdots \wedge \beta_l, \end{aligned} \quad (10)$$

where the hat  $\widehat{\phantom{x}}$  denotes the absence of the corresponding factor, and the brackets  $\{\alpha_i, \beta_j\}$  are given by the formula<sup>1</sup>:

$$\{\alpha, \beta\} = -\mathcal{D}_\alpha d\beta - \mathcal{D}_\beta d\alpha + d\mathcal{D}_\beta \alpha + [\alpha, d\beta]_\pi. \quad (11)$$

We call the bracket  $\{ , \}$  *Hawkins bracket*.

Hawkins showed that the Hawkins bracket satisfies the graded Jacobi identity,

$$\{\sigma, \{\tau, \rho\}\} - \{\{\sigma, \tau\}, \rho\} - (-1)^{\deg(\sigma)\deg(\tau)}\{\tau, \{\sigma, \rho\}\} = 0, \quad (12)$$

if and only if  $\mathcal{D}$  is flat and a certain 5-index tensor, called the metacurvature of  $\mathcal{D}$ , vanishes identically. In fact, Hawkins showed that if  $\mathcal{D}$  is flat, then it determines a (2,3)-type tensor field  $\mathcal{M}$  symmetric in the contravariant indices and antisymmetric in the covariant indices, given by

$$\mathcal{M}(df, \alpha, \beta) = \{f, \{\alpha, \beta\}\} - \{\{f, \alpha\}, \beta\} - \{\alpha, \{f, \beta\}\}. \quad (13)$$

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1. This formula appeared first in [1].

The tensor  $\mathcal{M}$  is the *metacurvature* of  $\mathcal{D}$ .

The following formulas, due to Hawkins, will be useful later. Let  $\alpha$  be a parallel 1-form; since  $\mathcal{D}$  is torsion-free,  $[\alpha, \eta]_\pi = \mathcal{D}_\alpha \eta$  for any  $\eta \in \Omega^*(M)$ , and so, by (11), the Hawkins bracket of  $\alpha$  and any 1-form  $\beta$  is given by

$$\{\alpha, \beta\} = -\mathcal{D}_\beta d\alpha. \quad (14)$$

Using this, one can deduce easily from (13) that for any parallel 1-forms  $\alpha, \beta$  and any 1-form  $\gamma$ ,

$$\mathcal{M}(\gamma, \beta, \alpha) = -\mathcal{D}_\gamma \mathcal{D}_\beta d\alpha. \quad (15)$$

### 2.3. The tensor $\mathbf{T}$

We now define the tensor  $\mathbf{T}$ , an essential ingredient in our main result.

Let  $(M, \pi)$  be a Poisson manifold endowed with a flat, torsion-free, contravariant  $\mathcal{F}^{reg}$ -connection  $\mathcal{D}$ . For each  $x \in M^{reg}$  and any  $a, b \in T_x^*M$ , define

$$\mathbf{T}_x(a, b) := \{\alpha, \beta\}(x) \quad (\in \wedge^2 T_x^*M), \quad (16)$$

where  $\{, \}$  denotes the Hawkins bracket associated to  $\mathcal{D}$ , and  $\alpha$  and  $\beta$  are parallel 1-forms defined in a neighborhood of  $x$  such that  $\alpha(x) = a$  and  $\beta(x) = b$ . (Such 1-forms exist, see Proposition 3.4.) This is independent of the choice of  $\alpha$  and  $\beta$  since by (14) and (6) we have

$$\mathbf{T}_x(a, b) = -(\mathcal{D}_\alpha d\beta)(x) = -(\mathcal{D}_\beta d\alpha)(x). \quad (17)$$

The assignment  $x \mapsto \mathbf{T}_x$  is then a smooth (2,2)-type tensor field on  $M^{reg}$ , symmetric in the contravariant indices and antisymmetric in the covariant indices, which by (15) verifies  $\mathcal{D}\mathbf{T} = \mathcal{M}$ , and which clearly vanishes if and only if the exterior differential of any parallel 1-form is also parallel.

### 3. Computation of the tensors $\mathcal{M}$ and $\mathbf{T}$

The metacurvature tensor is rather difficult to compute in general. In the symplectic case, Hawkins has established a simple formula for the metacurvature [5, Theorem 2.4]. Bahayou and the first author have also established in [1] a formula for the metacurvature in the case of a Lie-Poisson group endowed with a left-invariant Riemannian metric. In this section we explain how to compute the metacurvature (and the tensor  $\mathbf{T}$  as well), in the case of an  $\mathcal{F}^{reg}$ -connection, generalizing thus Hawkins's formula.

Throughout this section,  $\mathcal{D}$  will be a torsion-free contravariant  $\mathcal{F}^{reg}$ -connection on a  $d$ -dimensional Poisson manifold  $(M, \pi)$ .

We begin with the following simple lemma.

**Lemma 3.1.** *Let  $U \subseteq M$  be an open set on which the rank of  $\pi$  is constant. For any  $\alpha, \beta \in \Omega^1(U)$ ,  $\pi_{\sharp}(\beta) = 0$  implies  $\pi_{\sharp}(\mathcal{D}_{\alpha}\beta) = 0$ , and in this case,  $\mathcal{D}_{\alpha}\beta = \mathcal{L}_{\pi_{\sharp}(\alpha)}\beta$ .*

In other words, the kernel of the anchor map restricted to  $U$  is stable under  $\mathcal{D}$ . The next lemma shows that, around any regular point, there exists a complementary subbundle of  $\text{Ker } \pi_{\sharp}$  which is also stable under  $\mathcal{D}$ , provided that  $\mathcal{D}$  is flat.

**Lemma 3.2.** *If  $\mathcal{D}$  is flat, then for any  $x \in M^{reg}$  and any  $\mathcal{H}_0 \subseteq T_x^*M$  such that  $T_x^*M = (\text{Ker } \pi_{\sharp})_x \oplus \mathcal{H}_0$ , the cotangent bundle splits smoothly around  $x$  into:*

$$T^*M = (\text{Ker } \pi_{\sharp}) \oplus \mathcal{H}$$

with  $\mathcal{H}$  stable under  $\mathcal{D}$ , i.e.  $\mathcal{D}\mathcal{H} \subseteq \mathcal{H}$ , and  $\mathcal{H}_x = \mathcal{H}_0$ .

*Proof.* Let  $(U; x_i, y_u)$  ( $i = 1, \dots, 2r$ ;  $u = 1, \dots, d-2r$ ) be a local carte around  $x$  such that

$$\pi = \frac{1}{2} \sum_{i,j=1}^{2r} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

and the matrix  $(\pi_{ij})_{1 \leq i,j \leq 2r}$  is constant and invertible; let  $(\pi^{ij})_{1 \leq i,j \leq 2r}$  denote the inverse matrix. The restriction of  $\text{Ker } \pi_{\sharp}$  to  $U$  is a (rank  $d-2r$ ) subbundle of  $T|_U^*M$ , so we can choose a (arbitrary) smooth decomposition

$$T|_U^*M = (\text{Ker } \pi_{\sharp}) \oplus \mathcal{H}.$$

Then clearly  $\text{Ker } \pi_{\sharp} = \text{span}\{dy_u\}$ , and

$$\mathcal{H} = \text{span} \left\{ \theta_i = dx_i + \sum_{u=1}^{d-2r} B_i^u dy_u \right\}$$



for some functions  $B_i^u \in \mathcal{C}^\infty(U)$ . Since  $\mathcal{D}$  is a torsion-free  $\mathcal{F}$ -connection on  $U$ , one has  $\mathcal{D}_{dy_u} = \mathcal{D}dy_u = 0$  for all  $u$ . Thus, for any  $i, j$ ,

$$\begin{aligned}\mathcal{D}_{\theta_i}\theta_j &= \mathcal{D}_{dx_i}dx_j + \sum_{u=1}^{d-2r} \pi_{\sharp}(dx_i)(B_j^u) dy_u \\ &= \left( \sum_{k=1}^{2r} \Gamma_{ij}^k dx_k + \sum_{u=1}^{d-2r} \Gamma_{ij}^u dy_u \right) + \sum_{u=1}^{d-2r} \sum_{k=1}^{2r} \pi_{ik} \frac{\partial B_j^u}{\partial x_k} dy_u \\ &= \sum_{k=1}^{2r} \Gamma_{ij}^k \theta_k + \sum_{u=1}^{d-2r} \left( \Gamma_{ij}^u + \sum_{k=1}^{2r} \left( \pi_{ik} \frac{\partial B_j^u}{\partial x_k} - \Gamma_{ij}^k B_k^u \right) \right) dy_u,\end{aligned}$$

where  $\Gamma_{ij}^k, \Gamma_{ij}^u$  are the Christoffel symbols of  $\mathcal{D}$ . Therefore, the desired decomposition exists if and only if we may find a family of local functions  $\{B_i^u\}_{i,u}$  satisfying the following system of PDEs

$$\Gamma_{ij}^u + \sum_{k=1}^{2r} \left( \pi_{ik} \frac{\partial B_j^u}{\partial x_k} - \Gamma_{ij}^k B_k^u \right) = 0 \quad \forall i, j, \forall u,$$

or equivalently

$$\frac{\partial B_j^u}{\partial x_i} = \sum_{k=1}^{2r} \left( \sum_{l=1}^{2r} \pi^{il} \Gamma_{lj}^k \right) B_k^u - \sum_{l=1}^{2r} \pi^{il} \Gamma_{lj}^u \quad \forall i, j, \forall u. \quad (*)$$

In matrix notation, this is

$$\frac{\partial}{\partial x_i} B^u = \Gamma_i B^u + Y_i^u,$$

where

$$B^u = \begin{pmatrix} B_1^u \\ \vdots \\ B_{2r}^u \end{pmatrix}; \quad \Gamma_i = \left( \sum_{m=1}^{2r} \pi^{im} \Gamma_{mk}^l \right)_{1 \leq k, l \leq 2r}; \quad Y_i^u = - \sum_{j=1}^{2r} \pi^{ij} \begin{pmatrix} \Gamma_{j1}^u \\ \vdots \\ \Gamma_{j2r}^u \end{pmatrix}.$$

Considering the  $B_i^u$ 's as functions with variables  $x_i$  and parameters  $y_u$ , the system above can be solved, according to Frobenius's Theorem (see, e.g., [6, Theorem 1.1]), if and only if the integrability conditions

$$\Gamma_i \Gamma_j + \frac{\partial}{\partial x_j} \Gamma_i = \Gamma_j \Gamma_i + \frac{\partial}{\partial x_i} \Gamma_j, \quad \Gamma_i Y_j^u + \frac{\partial}{\partial x_j} Y_i^u = \Gamma_j Y_i^u + \frac{\partial}{\partial x_i} Y_j^u,$$

hold for all  $i, j$  and all  $u$ . With indices, these are respectively

$$\begin{aligned} \sum_{m=1}^{2r} \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m + \pi_{im} \frac{\partial \Gamma_{jk}^l}{\partial x_m} - \pi_{jm} \frac{\partial \Gamma_{ik}^l}{\partial x_m} &= 0, \\ \sum_{m=1}^{2r} \Gamma_{im}^u \Gamma_{jk}^m - \Gamma_{jm}^u \Gamma_{ik}^m + \pi_{im} \frac{\partial \Gamma_{jk}^u}{\partial x_m} - \pi_{jm} \frac{\partial \Gamma_{ik}^u}{\partial x_m} &= 0, \end{aligned}$$

which by (4) mean that the curvature vanishes. Thus (\*) has solutions (which depend smoothly on the parameters and the initial values).  $\square$

**Notation 3.3.** Given  $\mathcal{H}$  as above, the restriction of  $\pi_{\sharp}$  to  $\mathcal{H}$  defines an isomorphism from  $\mathcal{H}$  onto  $T\mathcal{S}$ ; we will denote by  $\varpi^{\mathcal{H}} : T\mathcal{S} \rightarrow \mathcal{H}$  its inverse.

**Proposition 3.4.** *The following are equivalent:*

- (a)  $\mathcal{D}$  is flat.
- (b) For any  $x \in M^{reg}$  and any  $a \in T_x^*M$ , there exists a 1-form  $\alpha$  defined in a neighborhood of  $x$  such that  $\alpha(x) = a$  and  $\mathcal{D}\alpha = 0$ .
- (c) Around any  $x \in M^{reg}$ , there exists a smooth coframe  $(\alpha_1, \dots, \alpha_d)$  of  $M$  such that  $\mathcal{D}\alpha_i = 0$  for all  $i$ . Such a coframe will be called flat.

*Proof.* The implications (b)  $\implies$  (c) and (c)  $\implies$  (a) are obvious. To show (a)  $\implies$  (b), let  $U \subseteq M$  be an open neighborhood of  $x$  on which the rank of  $\pi$  is constant. Over  $U$ ,  $T\mathcal{S}$  is a (involutive) regular distribution and  $\mathcal{D}$  is a torsion-free  $\mathcal{F}$ -connection. So we can define a partial connection  $\nabla$  on  $T|_U\mathcal{S}$  by setting for any  $\alpha, \beta \in \Omega^1(U)$ ,

$$\nabla_{\pi_{\sharp}(\alpha)} \pi_{\sharp}(\beta) = \pi_{\sharp}(\mathcal{D}_{\alpha}\beta). \quad (18)$$

One verifies immediately that the curvature tensor fields  $R^{\nabla}$  and  $R^{\mathcal{D}}$  respectively of  $\nabla$  and  $\mathcal{D}$  are related by:

$$R^{\nabla}(\pi_{\sharp}(a), \pi_{\sharp}(b))\pi_{\sharp}(c) = \pi_{\sharp}(R^{\mathcal{D}}(a, b)c) \quad \forall a, b, c \in T|_U^*M,$$

and hence  $R^{\nabla}$  vanishes since by hypothesis  $R^{\mathcal{D}}$  does. Using Frobenius's Theorem, we can then show in a way similar to the classical case that, for any  $v \in T_x\mathcal{S}$ , there exists a vector field  $X$  defined on some neighborhood of  $x$  such that  $X(x) = v$ ,  $X$  is tangent to  $T\mathcal{S}$ , that is,  $X(y) \in T_y\mathcal{S}$  for any  $y$  near  $x$ , and  $\nabla X = 0$ .

Now let  $a \in T_x^*M$ . According to Lemma 3.2, the cotangent bundle splits smoothly around  $x$  into:  $T^*M = (\text{Ker } \pi_\#) \oplus \mathcal{H}$  with  $\mathcal{H}$  stable under  $\mathcal{D}$ . Write  $a = b + c$  with  $b \in \text{Ker } \pi_\#(x)$  and  $c \in \mathcal{H}_x$ . By the argument above, there exists a  $\nabla$ -parallel vector field  $X$  defined in a neighborhood of  $x$  which is tangent to  $T\mathcal{S}$  and such that  $X(x) = \pi_\#(c)$ . Put  $\gamma = \varpi^\mathcal{H}(X) \in \Gamma(\mathcal{H})$ ; then  $\gamma(x) = c$ , and for any 1-form  $\phi$ ,  $\pi_\#(\mathcal{D}_\phi\gamma) = \nabla_{\pi_\#(\phi)}X = 0$  implying that  $\mathcal{D}\gamma = 0$ . Taking  $\alpha = \sum_{u=1}^s b_u dy_u + \gamma$ , where  $(y_u)$  is a family of local functions on  $M$  such that  $\text{Ker } \pi_\# = \text{span}\{dy_1, \dots, dy_s\}$  near  $x$ , and  $b_u$  are the coordinates of  $b$  in  $\{dy_1(x), \dots, dy_s(x)\}$ , we obtain finally the desired 1-form.  $\square$

The following corollary is a refinement of the preceding proposition.

**Corollary 3.5.** *If  $\mathcal{D}$  is flat, around any  $x \in M^{\text{reg}}$  there exists an  $\mathcal{S}$ -foliated coordinate system with leafwise coordinates  $\{x_i\}_{i=1}^{2r}$  and transverse coordinates  $\{y_u\}_{u=1}^{d-2r}$  such that for any  $\mathcal{H}$  as in Lemma 3.2,*

$$\mathbf{F}^* = (\phi_i := \varpi^\mathcal{H}(\partial/\partial x_i); dy_u)$$

*is a flat coframe of  $M$  near  $x$ . Such a coordinate system will be called flat.*

*Remark 3.6.* Another equivalent way of expressing that the  $\mathcal{S}$ -foliated coordinate system  $(x_i, y_u)$  is flat is the following:  $\nabla \partial/\partial x_i = 0$  for all  $i$ , where  $\nabla$  is the (local) partial connection defined by (18).

*We assume for the remainder of this section that  $\mathcal{D}$  is flat.*

We shall compute the tensors  $\mathcal{M}$  and  $\mathbf{T}$  in the coframe  $\mathbf{F}^*$ . To do so, we need first to determine its dual frame.

With the notations of Corollary 3.5, for each  $i$ , there exist unique functions,  $A_i^1, \dots, A_i^{d-2r}$ , defined in neighborhood of  $x$  such that

$$dx_i + \sum_{u=1}^{d-2r} A_i^u dy_u \in \mathcal{H}. \quad (19)$$

For any  $i$  and any  $u$  we put

$$X_i := -H_{x_i} = -\pi_\#(dx_i), \quad Y_u := \frac{\partial}{\partial y_u} - \sum_{i=1}^{2r} A_i^u \frac{\partial}{\partial x_i}. \quad (20)$$

**Lemma 3.7.** *With the above notations,  $(X_i, Y_u)$  is the dual frame to  $\mathbf{F}^*$ . Moreover, the vector fields  $X_i$  and  $Y_u$  are, respectively, Hamiltonian and Poisson, and verify*

$$\begin{aligned} [X_i, X_j] &= -\sum_{k=1}^{2r} \frac{\partial \pi_{ij}}{\partial x_k} X_k; \quad [X_i, Y_u] = \sum_{j=1}^{2r} \frac{\partial A_i^u}{\partial x_j} X_j; \\ [Y_u, Y_v] &= \sum_{i,j=1}^{2r} \pi^{ij} \left( \frac{\partial A_j^u}{\partial y_v} - \frac{\partial A_j^v}{\partial y_u} + \sum_{k=1}^{2r} A_k^u \frac{\partial A_j^v}{\partial x_k} - A_k^v \frac{\partial A_j^u}{\partial x_k} \right) X_i. \end{aligned} \quad (21)$$

Here,  $\pi_{ij} := \pi(dx_i, dx_j)$  and  $(\pi^{ij})$  is the inverse matrix of  $(\pi_{ij})$ .

*Proof.* The fact that  $(X_i, Y_u)$  is the dual frame to  $\mathbf{F}^*$  follows immediately, once we note that

$$\phi_i := \varpi^{\mathcal{H}}(\partial/\partial x_i) = \sum_{j=1}^{2r} \pi^{ij} \left( dx_j + \sum_{u=1}^{d-2r} A_j^u dy_u \right). \quad (22)$$

By definition, each of the vector fields  $X_i$  is Hamiltonian. To see that each  $Y_u$  is Poisson, observe that the equation  $[\phi_i, \phi_j]_{\pi} = 0$  yields

$$\begin{aligned} Y_u \cdot \pi(\phi_i, \phi_j) &= L_{\partial/\partial x_i} \phi_j(Y_u) - L_{\partial/\partial x_j} \phi_i(Y_u) \\ &= -\phi_j\left(\left[\frac{\partial}{\partial x_i}, Y_u\right]\right) + \phi_i\left(\left[\frac{\partial}{\partial x_j}, Y_u\right]\right) \\ &= -L_{Y_u} \phi_j\left(\frac{\partial}{\partial x_i}\right) + Y_u \cdot \pi(\phi_i, \phi_j) + L_{Y_u} \phi_i\left(\frac{\partial}{\partial x_j}\right) - Y_u \cdot \pi(\phi_j, \phi_i) \\ &= -\pi(\phi_i, L_{Y_u} \phi_j) - \pi(L_{Y_u} \phi_i, \phi_j) + 2Y_u \cdot \pi(\phi_i, \phi_j), \end{aligned}$$

hence  $L_{Y_u} \pi(\phi_i, \phi_j) = 0$ ; in addition, we have

$$L_{Y_u} \pi(\phi_i, dy_v) = -\pi(\phi_i, L_{Y_u} dy_v) = -\pi(\phi_i, d(Y_u(y_v))) = 0,$$

and it is clear that we also have  $L_{Y_u} \pi(dy_v, dy_w) = 0$ . It follows that  $L_{Y_u} \pi = 0$ , which means that  $Y_u$  is Poisson. Finally,

$$[X_i, X_j] = H_{\pi(dx_i, dx_j)} = -\sum_{k=1}^{2r} \frac{\partial \pi_{ij}}{\partial x_k} X_k, \quad [X_i, Y_u] = H_{Y_u(x_i)} = \sum_{j=1}^{2r} \frac{\partial A_i^u}{\partial x_j} X_j,$$

and the last equality of (21) follows by direct computation.  $\square$

We now can give the expression of the metacurvature in the coframe  $\mathbf{F}^*$ .

**Theorem 3.8.** *With the same notations as above, we have*

- (a) *For any  $u = 1, \dots, d - 2r$ ,  $\mathcal{M}(dy_u, \cdot, \cdot) = 0$ .*
- (b) *For any  $i, j, k = 1, \dots, 2r$ ,*

$$\begin{aligned} \mathcal{M}(\phi_i, \phi_j, \phi_k) &= - \sum_{l < m} \frac{\partial^3 \pi_{lm}}{\partial x_i \partial x_j \partial x_k} \phi_l \wedge \phi_m + \sum_{l, u} \frac{\partial^3 A_l^u}{\partial x_i \partial x_j \partial x_k} \phi_l \wedge dy_u \\ &+ \sum_{u < v, l} \frac{\partial^2}{\partial x_i \partial x_j} \left( \pi^{kl} \left( \frac{\partial A_l^u}{\partial y_v} - \frac{\partial A_l^v}{\partial y_u} + \sum_m A_m^u \frac{\partial A_l^v}{\partial x_m} - A_m^v \frac{\partial A_l^u}{\partial x_m} \right) \right) dy_u \wedge dy_v. \end{aligned}$$

*Proof.* Part (a) is immediate from (13) and (9).

For (b), on the one hand, we have by (15),

$$\mathcal{M}(\phi_i, \phi_j, \phi_k) = -\mathcal{D}_{\phi_i} \mathcal{D}_{\phi_j} d\phi_k \quad \text{for all } i, j, k.$$

On the other hand, using Lemma 3.7 gives

$$\begin{aligned} d\phi_i &= \sum_{j < k} \frac{\partial \pi_{jk}}{\partial x_i} \phi_j \wedge \phi_k - \sum_{j, u} \frac{\partial A_j^u}{\partial x_i} \phi_j \wedge dy_u \\ &- \sum_{u < v, j} \pi^{ij} \left( \frac{\partial A_j^u}{\partial y_v} - \frac{\partial A_j^v}{\partial y_u} + \sum_k A_k^u \frac{\partial A_j^v}{\partial x_k} - A_k^v \frac{\partial A_j^u}{\partial x_k} \right) dy_u \wedge dy_v, \end{aligned} \tag{23}$$

and the desired formula follows.  $\square$

Likewise, we get the following expression for the tensor  $\mathbf{T}$ .

- Theorem 3.9.** (i) *For any  $u = 1, \dots, d - 2r$ ,  $\mathbf{T}(dy_u, \cdot) = 0$ .*  
(ii) *For any  $i, j, k = 1, \dots, 2r$ ,*

$$\begin{aligned} \mathbf{T}(\phi_i, \phi_j) &= - \sum_{k < l} \frac{\partial^2 \pi_{kl}}{\partial x_i \partial x_j} \phi_k \wedge \phi_l + \sum_{k, u} \frac{\partial^2 A_k^u}{\partial x_i \partial x_j} \phi_k \wedge dy_u \\ &+ \sum_{u < v, k} \frac{\partial}{\partial x_i} \left( \pi^{jk} \left( \frac{\partial A_k^u}{\partial y_v} - \frac{\partial A_k^v}{\partial y_u} + \sum_l A_l^u \frac{\partial A_k^v}{\partial x_l} - A_l^v \frac{\partial A_k^u}{\partial x_l} \right) \right) dy_u \wedge dy_v. \end{aligned}$$

### 3.1. The symplectic case

If the Poisson tensor  $\pi$  is invertible, then the flat and torsion-free contravariant connection  $\mathcal{D}$  is an  $\mathcal{F}$ -connection<sup>2</sup>, and is related to a flat, torsion-free, covariant connection  $\nabla$  on  $M$  via  $\pi_{\sharp}(\mathcal{D}_{\alpha}\beta) = \nabla_{\pi_{\sharp}(\alpha)}\pi_{\sharp}(\beta)$ . In that case, a flat coordinate system is one with respect to whom  $\nabla$  is given trivially by partial derivatives (Remark 3.6).

Since the kernel of the anchor map reduces to zero, the metacurvature vanishes if and only if  $\pi$  is quadratic in the affine structure defined by  $\nabla$  (Theorem 3.8), which is precisely the conclusion of [5, Theorem 2.4].

Likewise, the tensor  $\mathbf{T}$  vanishes if and only if the components of  $\pi$  w.r.t. any flat coordinate system are at most of degree one (Theorem 3.9).

**Example 3.10.** If  $\mathcal{D}$  is a flat, torsion-free, Poisson connection on a Poisson manifold  $(M, \pi)$  with  $\pi$  invertible, then  $\mathbf{T}$  vanishes identically. In fact, the condition  $\mathcal{D}\pi = 0$  is equivalent to saying that the components of  $\pi$  with respect to any flat coordinate system are constant.

**Example 3.11.** Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ , and let  $r \in \wedge^2 \mathfrak{g}$  be a solution of the classical Yang-Baxter equation. For any tensor  $\tau$  on  $\mathfrak{g}$ , denote by  $\tau^+$  the corresponding left-invariant tensor field on  $G$ . Following [11], the formula

$$\mathcal{D}_{a^+}^r b^+ = -(\text{ad}_{r(a)}^* b)^+,$$

where  $a, b \in \mathfrak{g}^*$ , defines a left-invariant, flat, torsion-free,  $\mathcal{F}$ -connection  $\mathcal{D}^r$  on  $(G, r^+)$  with vanishing  $\mathbf{T}$ . It is well known (see, e.g., [2]) that if  $r$  is invertible, then the left-invariant symplectic form  $\omega^+$  inverse of  $r^+$  defines a left-invariant, flat, torsion-free connection  $\nabla$  on  $G$  via

$$\omega^+(\nabla_{u^+} v^+, w^+) = -\omega^+(v^+, [u^+, w^+]), \quad u, v, w \in \mathfrak{g}.$$

One then checks easily that  $\mathcal{D}^r$  and  $\nabla$  are related by:  $r_{\sharp}^+(\mathcal{D}_{a^+}^r b^+) = \nabla_{r(a)+r(b)}^+ b^+$ . We thus recover a result of the first author and Medina (cf. [12, Theorem 1.1-(1)]) which states that if  $r$  is invertible, then  $r^+$  is polynomial of degree at most 1 with respect to the affine structure defined by  $\nabla$ .

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2. Actually, this is true for any contravariant connection on  $M$  since  $\text{Ker } \pi_{\sharp} = \{0\}$ .

### 3.2. The Riemannian case

Let  $\mathcal{D}$  be the metric contravariant connection associated to a Poisson tensor  $\pi$  and a Riemannian metric  $g$  on a manifold  $M$ . Thanks to the metric  $g$ , the cotangent bundle splits orthogonally into

$$T^*M = \text{Ker } \pi_{\sharp} \oplus (\text{Ker } \pi_{\sharp})^{\perp}.$$

**Lemma 3.12.** *Let  $U \subseteq M$  be an open set on which the rank of  $\pi$  is constant. Assume that  $\mathcal{D}$  is an  $\mathcal{F}$ -connection on  $U$ . Then  $(\text{Ker } \pi_{\sharp}|_U)^{\perp}$  is stable under  $\mathcal{D}$ .*

Thus if  $\mathcal{D}$  is flat and an  $\mathcal{F}^{reg}$ -connection, then by Corollary 3.5 there exists around any  $x \in M^{reg}$  an  $\mathcal{S}$ -foliated chart with leafwise coordinates  $\{x_i\}_{i=1}^{2r}$  and transverse coordinates  $\{y_u\}_{u=1}^{d-2r}$  such that  $\{\phi_i := \varpi^{\perp}(\partial/\partial x_i); dy_u\}$  is a flat coframe of  $M$  near  $x$ , where we have denoted by  $\varpi^{\perp} : T\mathcal{S} \rightarrow (\text{Ker } \pi_{\sharp})^{\perp}$  the inverse of  $\pi_{\sharp} : (\text{Ker } \pi_{\sharp})^{\perp} \rightarrow T\mathcal{S}$ . In this case, the functions  $A_i^u$  defined by (19) can be computed by means of the metric; indeed, using (22) and the fact that  $\langle \phi_i, dy_u \rangle = 0$ , one has  $-A_i^u = \sum_v g_{iv} g^{uv}$  where  $g_{iv} = \langle dx_i, dy_v \rangle$  and  $(g^{uv})$  is the inverse matrix of the one whose coefficients are  $g_{uv} = \langle dy_u, dy_v \rangle$ .

## 4. Proof of Theorem 1.1

Let  $(x_i, y_u)$ , with  $i = 1, \dots, 2r$  and  $u = 1, \dots, d - 2r$ , be a flat coordinate system around  $x_0$ , choose  $\mathcal{H}$  as in lemma 3.2, and let  $\mathbf{F}^* = \{\phi_i, dy_u\}$  be the corresponding flat coframe and  $\{X_i, Y_u\}$  its dual frame. We shall construct a family of vector fields  $\{Z_1, \dots, Z_{2r}\}$  on a neighborhood  $U$  of  $x_0$  which span  $T\mathcal{S}$  and commute with the  $X_i$ 's and the  $Y_u$ 's. In that case,

- The family  $\{Z_1, \dots, Z_{2r}\}$  will form a  $2r$ -dimensional real Lie algebra  $\mathfrak{g}$ , since by the Jacobi identity

$$[[Z_i, Z_j], X_l] = [[Z_i, Z_j], Y_u] = 0 \quad \forall i, j, l, \forall u,$$

so that  $[Z_i, Z_j] = \sum_k c_{ij}^k Z_k$  with  $c_{ij}^k$  being constant; it is then clear that  $\mathfrak{g}$  acts freely on  $U$ .

- The Poisson tensor  $\pi$  will be expressed as

$$\pi = \frac{1}{2} \sum_{i,j} a_{ij} Z_i \wedge Z_j$$

where the matrix  $(a_{ij})_{1 \leq i,j \leq 2r}$  is constant and invertible: since the  $X_i$ 's and the  $Y_u$ 's are Poisson (Lemma 3.7), then writing  $\pi = \sum_{i < j} a_{ij} Z_i \wedge Z_j$  where  $a_{ij} \in \mathcal{C}^{\infty}(U)$ , we get  $X_k(a_{ij}) = Y_u(a_{ij}) = 0$ .

– The connection  $\mathcal{D}$  will be given on  $U$  by

$$\mathcal{D}_\alpha \beta = \sum_{i,j} a_{ij} \alpha(Z_i) \mathcal{L}_{Z_j} \beta.$$

In fact, this is true for any  $\beta \in \mathbf{F}^*$  since  $\mathcal{L}_{Z_i} \phi_j = \mathcal{L}_{Z_i} dy_u = 0$ , and  $\mathcal{D}_\alpha \beta - \sum_{i,j} a_{ij} \alpha(Z_i) \mathcal{L}_{Z_j} \beta$  is tensorial in  $\beta$  as  $\pi_\#(\alpha) = \sum_{i,j} a_{ij} \alpha(Z_i) Z_j$ .

We shall proceed in two steps. We first construct a family of vector fields which span  $T\mathcal{S}$  and commute with the  $X_i$ 's, and then construct from this the desired family.

To start, observe that by virtue of Theorem 1 and Lemme 3.7 we have

$$[X_i, X_j] = \sum_{k=1}^{2r} \lambda_{ij}^k X_k, \quad [X_i, Y_u] = \sum_{j=1}^{2r} \mu_{iu}^j X_j, \quad [Y_u, Y_v] = \sum_{i=1}^{2r} \nu_{uv}^i X_i,$$

where  $\lambda_{ij}^k, \mu_{iu}^j, \nu_{uv}^i$  are Casimir functions. Let  $\mathcal{T} \subseteq M$  be a smooth transversal to  $T\mathcal{S}$  intersecting  $x_0$ ; this is parametrized by the  $y_u$ 's. Fixing  $y \in \mathcal{T}$ , the restrictions  $X_1^y, \dots, X_{2r}^y$  of  $X_1, \dots, X_{2r}$  to the symplectic leaf  $\mathcal{S}_y$  passing through  $y$  form a Lie algebra  $\mathfrak{g}_y$  which acts freely and transitively on  $\mathcal{S}_y$ . Therefore, according to [3], there exists a free transitive Lie algebra anti-homomorphism  $\hat{\Gamma}_y : \mathfrak{g}_y \rightarrow \mathfrak{X}^1(\mathcal{S}_y)$  whose image is

$$\hat{\Gamma}_y(\mathfrak{g}_y) = \{T \in \mathfrak{X}^1(\mathcal{S}_y) : [T, X_i^y] = 0 \ \forall i = 1, \dots, 2r\},$$

and such that  $\hat{\Gamma}_y(X_i^y)(y) = X_i(y)$  for all  $i$ . Setting for any  $i$ ,

$$T_i(z) := \hat{\Gamma}_y(X_i^y)(z), \quad z \in \mathcal{S}_y$$

and varying  $y$  along  $\mathcal{T}$ , we get a family of linearly independent vector fields  $\{T_1, \dots, T_{2r}\}$  which are tangent to  $T\mathcal{S}$  and verify

$$[T_i, X_j] = 0 \quad \text{for all } i, j,$$

and such that  $T_i(y) = X_i(y)$  for all  $i$  and all  $y \in \mathcal{T}$ . Note that  $T_1, \dots, T_{2r}$  are smooth since the solutions of the system

$$[T, X_i] = 0, \quad i = 1, \dots, 2r$$

depend smoothly on the parameter  $y \in \mathcal{T}$  and the initial values along  $\mathcal{T}$ . It is also worth noting that since the  $\mu_{iu}^j$ 's are Casimir, we have

$$[X_i, [T_j, Y_u]] = 0 \quad \text{for all } i, j \text{ and all } u,$$



so that

$$[T_i, Y_u] = \sum_{j=1}^{2r} \gamma_{iu}^j T_j,$$

where  $\gamma_{iu}^j$  are Casimir functions; in addition, since the  $\nu_{uv}^i$ 's are Casimir, we have

$$[T_i, [Y_u, Y_v]] = 0 \quad \text{for all } i \text{ and all } u, v$$

implying

$$\frac{\partial \gamma_{ju}^i}{\partial y_v} - \frac{\partial \gamma_{jv}^i}{\partial y_u} + \sum_{k=1}^{2r} \gamma_{ku}^i \gamma_{jv}^k - \gamma_{kv}^i \gamma_{ju}^k = 0 \quad (*)$$

for all  $i, j$  and all  $u, v$ .

Now we would like to find an invertible matrix  $\xi = (\xi_{ij})_{1 \leq i, j \leq 2r}$  where  $\xi_{ij}$  are Casimir functions such that the vector fields

$$Z_i := \sum_{j=1}^{2r} \xi_{ji} T_j, \quad i = 1, \dots, 2r$$

verify

$$[Z_i, Y_u] = 0 \quad \text{for all } i \text{ and all } u.$$

If such a matrix exists, the family  $\{Z_1, \dots, Z_{2r}\}$  is clearly the desired one. Since the functions  $\xi_{ij}$  are searched to be Casimir, the condition for the  $Z_i$ 's to commute with the  $Y_u$ 's can be rewritten as

$$\frac{\partial \xi_{ji}}{\partial y_u} = \sum_{k=1}^{2r} \gamma_{ku}^j \xi_{ki} \quad \forall i, j, \forall u,$$

or in matrix notation

$$\frac{\partial}{\partial y_u} \xi_i = \Gamma_u \xi_i,$$

where  $\xi_1, \dots, \xi_{2r}$  are the colon row of  $\xi$  and  $\Gamma_u := (\gamma_{ju}^i)_{1 \leq i, j \leq 2r}$ . So we need to solve this system. Again, since the functions  $\xi_{ij}$  are searched to be Casimir, we can solve it on  $\mathcal{T}$ . According to Frobenius's Theorem, this system has solutions if and only if the following integrability condition

$$\Gamma_u \Gamma_v + \frac{\partial}{\partial y_v} \Gamma_u = \Gamma_v \Gamma_u + \frac{\partial}{\partial y_u} \Gamma_v$$

holds for all  $u, v$ , which is nothing else but  $(*)$ . It then suffices to take  $\xi_{ij}(x_0) = \delta_{ij}$  as initial conditions to conclude. Finally, if  $\mathcal{D}$  is the metric contravariant connection with respect to  $\pi$  and a Riemannian metric  $g$ , we choose  $\mathcal{H} = (\text{Ker } \pi_{\sharp})^{\perp}$ . In this case, we have

$$\mathcal{L}_{Z_i} g(\phi_j, \phi_k) = \mathcal{L}_{Z_i} g(\phi_j, dy_u) = \mathcal{L}_{Z_i} g(dy_u, dy_v) = 0$$

since  $\mathcal{L}_{Z_i} \phi_j = \mathcal{L}_{Z_i} dy_u = 0$  and since  $g(\phi_i, \phi_j)$  and  $g(dy_u, dy_v)$  are Casimir functions. This shows that the vector fields  $Z_i$  are Killing.  $\square$

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